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Eddy-damped quasinormal Markovian closure: a closure for magnetohydrodynamic turbulence?

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Abstract

A set of nonlinear differential equations are developed that are analogous to the spectral evolution equations of incompressible magnetohydrodynamics (MHD). Because these equations possess little detail of MHD, apart from salient symmetry properties, they provide a toy model in which aspects of turbulent MHD can be understood readily. In the context of this model, the eddy-damped quasinormal Markovian (EDQNM) closure often used in Navier–Stokes turbulence is demonstrated to provide physically realizable spectra for magnetohydrodynamic turbulence, if the eddy-damping functions are chosen to satisfy certain symmetry properties. The requirements of physical realizability are more demanding in MHD than in fluid turbulence. In the absence of mean fields, this model demonstrates that the components of not only the turbulent kinetic energy spectrum, but also the magnetic energy spectra, never become negative. Another condition for realizability possessed by this model is that the components of the turbulent cross-helicity spectrum always satisfy a Schwarz inequality with respect to the corresponding components of the kinetic and magnetic energy spectra.

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1. Introduction

The concept of a turbulent magnetohydrodynamic dynamo that sustains mean magnetic fields against resistive diffusion has permeated both the astrophysical and fusion literature ever since Elsasser introduced the idea in the context of the Earth's geomagnetic field [1]. The term 'turbulent' suggests that an appropriate method of analysing this dynamo phenomenon would be to use the theoretical apparatus that has been employed with varying degrees of success in the description of turbulent fluids. In particular, we wish to develop a theoretical description of the evolution of the statistical behaviour of a physical turbulent magnetofluid. A particular application that has been of interest is the reversed-field pinch [2]. To arrive at this goal, various hurdles need to be overcome.

First, the effects of loss mechanisms, such as resistivity and viscosity, must be included in such a description. During the 1970s the concept of an absolute equilibrium ensemble for the description of an ideal (i.e., lossless) turbulent magnetofluid was utilized [3]. Although such analyses had the potential for contributing a broader understanding of the reversed-field pinch than was afforded by Taylor's hypothesis [4], the presence of high-wavenumber equipartition—the standard Rayleigh–Jeans pathology of a classical equilibrium system—prevented its realization. The total turbulent magnetic energy was divergent; two-point $\boldsymbol{B}$ correlations, $\langle \boldsymbol{B}(\boldsymbol{r})\boldsymbol{B}(\boldsymbol{r}')\rangle$, possessed the term, $\delta(\boldsymbol{r}-\boldsymbol{r}')$. [5]. To avoid this divergence in this case of classical dynamics of continua, the presence of the loss mechanisms of resistivity and viscosity that damp out the high-wavenumber spectrum must be taken into account.

Second, we need to be able to describe the evolution of inhomogeneous turbulence because the reversed-field pinch is a bounded system. There has been scant literature relevant to the application of formal closures to inhomogeneous fluid turbulence. However, one of us (LT) has adapted his spectral expansion of solenoidal fields in a bounded magnetofluid [5] to the description of the evolution of a bounded, incompressible, turbulent Navier–Stokes fluid [6]. This adaptation depended on the use of a random-phase approximation, an approximation that has been supported by direct numerical simulations in the absence of any mean flow [7].

Third, we need to verify that our closure satisfies the most basic physical principles: positivity of the components of both the turbulent magnetic energy spectrum and the turbulent kinetic energy spectrum, as well as a Schwarz inequality for components of the cross-helicity spectrum. If the spectra satisfy these minimum requirements, the closure is said to provide realizable spectra. Verifying such satisfaction for the case of MHD turbulence demands a more intricate analysis than was required for the case of purely Navier–Stokes turbulence in which only a kinetic energy spectrum is involved [8]. This analysis is the subject of this paper. A related earlier analysis of Pouquet *et al* [9] sets the cross-helicity at zero and then implements the eddy-damped quasinormal Markovian (EDQNM) closure for a spatially homogeneous MHD turbulence.

Fourth, we eventually shall need to be able to describe the turbulent dynamics when nontrivial mean fields are present, such as the mean reversed magnetic field of the reversed-field pinch. The fact that we have obtained a mean reversed magnetic field when we utilized an absolute equilibrium ensemble description of a turbulent ideal magnetofluid, provides the main impetus behind our goal of developing a statistical understanding of turbulent magnetohydrodynamics in the presence of resistivity and viscosity. We would like to demonstrate that the reversed mean magnetic field can be sustained when the system is tweaked with dissipation in order to remove the Rayleigh–Jeans ultraviolet catastrophe induced by an equilibrium's high-wavenumber equipartition. In the case of inhomogeneous Navier–Stokes turbulence, we have seen how a turbulent energy spectrum that is not reflection-invariant can induce a mean flow [10]. This is reminiscent of the necessity for a parity violation in the turbulence for the development of a mean magnetic field [11]. However, to treat turbulence with such mean fields, we shall need to learn how to adapt and utilize an appropriate closure.

2. Schema of the Elsasser equations and invariants for incompressible magnetohydrodynamics

The fundamental equations of incompressible magnetohydrodynamics are:

$$\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t}+\boldsymbol{v}(\boldsymbol{r},t)\cdot\nabla\boldsymbol{v}(\boldsymbol{r},t)=-\nabla\tilde{p}(\boldsymbol{r},t)+[\nabla\times\boldsymbol{B}(\boldsymbol{r},t)]\times\boldsymbol{B}(\boldsymbol{r},t)+\nu\nabla^2\boldsymbol{v}(\boldsymbol{r},t) \qquad (1)$$

$$\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} = \nabla \times [\boldsymbol{v}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t)] + \eta \nabla^2 \boldsymbol{B}(\boldsymbol{r},t) \tag{2}$$

$$\nabla \cdot \boldsymbol{v}(\boldsymbol{r},t) = \nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0. \tag{3}$$

In these equations, $\boldsymbol{v}$ and $\boldsymbol{B}$ respectively represent the fluid velocity and the Alfven velocity, a quantity directly proportional to the magnetic field. The quantity $\tilde{p}$ represents the pressure divided by the mass density. The parameters ν and η represent the kinematic viscosity and resistivity, respectively.

If one replaces the velocity and magnetic field variables, $\boldsymbol{v}$ and $\boldsymbol{B}$, by the Elsasser variables, $\boldsymbol{W}_+$ and $\boldsymbol{W}_-$, where

$$\boldsymbol{W}_+ = \frac{\boldsymbol{v}+\boldsymbol{B}}{2} \qquad \boldsymbol{W}_- = \frac{\boldsymbol{v}-\boldsymbol{B}}{2} \qquad \nabla \cdot \boldsymbol{W}_\pm = 0 \tag{4}$$

we obtain the following equation describing the evolution of their curls, $\vec{\omega}_\pm \equiv \nabla \times \boldsymbol{W}_\pm$:

$$\begin{aligned} &\frac{\partial \vec{\omega}_\pm}{\partial t} - \gamma_2 \nabla^2 \vec{\omega}_\pm - \gamma_1 \nabla^2 \vec{\omega}_\mp \\ &\quad = \nabla \times (\boldsymbol{W}_\pm \times \vec{\omega}_\mp) + \nabla \times (\boldsymbol{W}_\mp \times \vec{\omega}_\pm) + \nabla \times \nabla \times (\boldsymbol{W}_\mp \times \boldsymbol{W}_\pm) \end{aligned} \tag{5}$$

where

$$\gamma_i = \frac{\nu + (-)^i \eta}{2} \qquad i = 1, 2. \tag{6}$$

Note that we can define the ideal MHD invariants: magnetic energy by $\frac{1}{2}E^B \equiv \frac{1}{2}\int B^2\,\mathrm{d}^3r$, the kinetic energy by $\frac{1}{2}E^K \equiv \frac{1}{2}\int v^2\,\mathrm{d}^3r$ and the cross-helicity by $H^C \equiv \int \boldsymbol{v}\cdot\boldsymbol{B}\,\mathrm{d}^3r$ [11]. Then

$$\begin{aligned} E^B &= \int \left(W_+^2 + W_-^2 - 2\boldsymbol{W}_+ \cdot \boldsymbol{W}_-\right) \mathrm{d}^3r \\ E^K &= \int \left(W_+^2 + W_-^2 + 2\boldsymbol{W}_+ \cdot \boldsymbol{W}_-\right) \mathrm{d}^3r \\ H^C &= \int \left(W_+^2 - W_-^2\right) \mathrm{d}^3r. \end{aligned} \tag{7}$$

At this point, we could choose a specific geometry and set of boundary conditions; that would lead us to a choice of an orthonormal solenoidal expansion basis for representing the incompressible velocity and magnetic fields. For simplicity, we shall assume that both fields satisfy the same boundary conditions and thus can be expanded in the same set of orthonormal, solenoidal basis vectors, labelled by the appropriate triplet of numbers, which we shall denote as $\boldsymbol{k}$. (Examples would be a solenoidal basis satisfying periodic boundary conditions or, as appropriate for bounded domains, a Chandrasekhar–Kendall basis formed from eigenvectors of the curl operator [12]. We thus let

$$\boldsymbol{W}_+(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} X_{\boldsymbol{k}}(t)\,\vec{\xi}_{\boldsymbol{k}}(\boldsymbol{r}) \qquad \boldsymbol{W}_-(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} Y_{\boldsymbol{k}}(t)\,\vec{\xi}_{\boldsymbol{k}}(\boldsymbol{r}). \tag{8}$$

To emphasize the salient structure of these equations, a structure that will be seen to permit realizability of the turbulent spectra under the closure, we make a further simplification. We employ the time-honoured technique introduced by Herring [13]. We shall denote the triplet of numbers, $\boldsymbol{k}$, with a single scalar index, i, and let it vary from $-N$ to N. This is the analogue of letting sums vary from $-\boldsymbol{k}$ to $+\boldsymbol{k}$. Similarly, many of the symmetry conditions that we shall be imposing on various functions of these scalar indices are analogous to those that occur on corresponding functions of wave-vector indices [6].

By virtue of the symmetric structure of the Elsasser equations, we find the following structure for the evolution of the spectral coefficients, $X_i(t)$ and $Y_i(t)$:

$$\begin{aligned}\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} + \alpha_i X_i(t) + \beta_i Y_i(t) &= \sum_{j,k=-N}^{N} c_{jki} X_j(t) Y_k(t) \\ \frac{\mathrm{d}Y_i(t)}{\mathrm{d}t} + \alpha_i Y_i(t) + \beta_i X_i(t) &= \sum_{j,k=-N}^{N} c_{jki} Y_j(t) X_k(t).\end{aligned} \tag{9}$$

The coefficients, α_i and β_i, represent dissipation coefficients. For simplicity, all of the quantities in equations (9) are taken to be real.

One should note from equation (7) that when dissipation is absent, conservation of both the total energy, $E^B + E^K$, and the cross-helicity, H^C, is entirely equivalent to conservation of $\int W_+^2\,\mathrm{d}^3r$ and $\int W_-^2\,\mathrm{d}^3r$. Because of the assumed orthonormality of the set of basis functions, $\{\vec{\xi}_{\boldsymbol{k}}(\boldsymbol{r})\}$, we note that this conservation implies that when $\alpha = 0$ and $\beta = 0$,

$$\sum_{i=-N}^{N} \frac{\mathrm{d}[X_i(t)]^2}{\mathrm{d}t} = \sum_{i=-N}^{N} \frac{\mathrm{d}[Y_i(t)]^2}{\mathrm{d}t} = 0. \tag{10}$$

From equations (9), we note then that conservation of the ideal MHD invariants of energy and cross-helicity imposes the following symmetry condition on the coupling coefficients:

$$c_{jki} + c_{ikj} = 0. \tag{11}$$

Equations (9) and (11) constitute a toy model, which we shall investigate, that possesses those salient features of the actual magnetohydrodynamic equations needed to assess the ability of a turbulent closure (the eddy-damped quasinormal Markovian (EDQNM) closure) to preserve the physical realizability of the turbulent spectra. At the same time, equations (9) dispense with much of the additional dynamical and geometrical information that would be contained in a faithful representation of MHD physics, but which would also obscure the necessary requirements of a closure for yielding physically realizable turbulent spectra.

We are not introducing our toy model to ignore any of the essential issues relating to the utility of the EDQNM closure for magnetohydrodynamic turbulence that contains cross-helicity. Instead, we are introducing our model in the same spirit with which we are using Herring's stripped-down notation; i.e., because it is stripped of irrelevant details. An analysis employing the complete MHD equations using a complete set of appropriate solenoidal basis functions for their representation would result in unnecessarily obscuring the concepts and would be relatively unwieldy. Our toy model, with its distillation of the essential physics and symmetries, 'helps the medicine go down.' For examples of lengthy precursor analyses see [6].

3. Requirements for realizability of the spectra

Let us define turbulent spectra through the following ensemble averages:

$$U_i^x(t) \equiv \langle X_i(t) X_i(t)\rangle \qquad U_i^y(t) \equiv \langle Y_i(t) Y_i(t)\rangle \qquad U_i^z(t) \equiv \langle X_i(t) Y_i(t)\rangle. \tag{12}$$

From their definition alone, they must satisfy the following two positivity conditions at all times:

$$U_i^x(t) \geqslant 0 \qquad U_i^y(t) \geqslant 0 \qquad \forall\, i. \tag{13}$$

Also for all time, the following Schwarz inequality must be satisfied:

$$\left|U_i^z(t)\right| \leqslant \left[U_i^x(t) U_i^y(t)\right]^{\frac{1}{2}} \qquad \forall\, i. \tag{14}$$

From equations (7), we see that the spectral component of the magnetic energy is given by

$$E_i^B = U_i^x + U_i^y - 2U_i^z. \tag{15}$$

The corresponding spectral component of the kinetic energy is given by

$$E_i^K = U_i^x + U_i^y + 2U_i^z. \tag{16}$$

Finally, the spectral component of the cross-helicity is given by

$$H_i^C = U_i^x - U_i^y. \tag{17}$$

Thus knowledge of the spectra of U^x, U^y and U^z specifies the magnetic energy, kinetic energy and cross-helicity spectra.

We shall show now that equations (13) and (14) guarantee that E_i^B and E_i^K are never negative, and that $|H_i^C| \leqslant (E_i^B E_i^K)^{\frac{1}{2}}$. First, we observe from equations (13), (15) and (16) that E_i^B and E_i^K will each be non-negative if

$$|U_i^z| \leqslant \frac{U_i^x + U_i^y}{2}. \tag{18}$$

But clearly,

$$\frac{\left[(U_i^x)^{\frac{1}{2}} \pm (U_i^y)^{\frac{1}{2}}\right]^2}{2} = \left(\frac{U_i^x + U_i^y}{2}\right) \pm (U_i^x U_i^y)^{\frac{1}{2}} \geqslant 0. \tag{19}$$

With this result and equations (13) and (14), we observe that indeed

$$|U_i^z| \leqslant (U_i^x U_i^y)^{\frac{1}{2}} \leqslant \frac{U_i^x + U_i^y}{2} \tag{20}$$

guaranteeing that E_i^B and E_i^K are not negative.

We next demonstrate that equations (13) and (14) also guarantee that $|H_i^C| \leqslant (E_i^B E_i^K)^{\frac{1}{2}}$. We first note from the definitions, equations (4) and (8), that

$$U_i^x = \left\langle \frac{(v_i + B_i)^2}{4} \right\rangle \qquad U_i^y = \left\langle \frac{(v_i - B_i)^2}{4} \right\rangle \qquad U_i^z = \left\langle \frac{(v_i^2 - B_i^2)}{4} \right\rangle. \tag{21}$$

Then, using equations (13) and (14) we obtain

$$(U_i^z)^2 \leqslant U_i^x U_i^y \tag{22}$$

or, equivalently,

$$\left\langle (v_i^2 - B_i^2)^2 \right\rangle \leqslant \left\langle (v_i^2 + B_i^2)^2 - 4(v_i B_i)^2 \right\rangle \tag{23}$$

which is tantamount to

$$(H_i^c)^2 = \langle (v_i B_i)^2 \rangle \leqslant \langle v_i^2 \rangle \langle B_i^2 \rangle = E_i^B E_i^K \tag{24}$$

the Schwarz inequality satisfied by elements of the cross-helicity spectrum, H_i^C.

4. The random phase approximation and the EDQNM closure

The physics of turbulence is not an exact science. There are no closed sets of equations. The fundamental equation of Newtonian fluids is the Navier–Stokes equation. Its extension to electrically conducting fluids and incorporating the Maxwell equations, normally ignoring the displacement current term, is the set of equations that governs magnetohydrodynamics. The physical quantities of interest in turbulence are often statistical ones. The attempt at application of the Navier–Stokes or MHD equations to the description of turbulent fluids therefore entails taking averages. By virtue of the nonlinearity of the equations, the evolution of a mean field is governed by the mean value of the product of two fields; the evolution of the mean value of the product of the two fields is governed by the mean value of the product of three fields, etc. A closure, such as EDQNM, entails the use of some plausible assumption that results in replacing the infinite chain of equations by a finite set of equations. Most equations used in physics are closures. In order to describe fluid motion, we do not solve an Avogadro number of Newton's equations describing the evolution of the discrete particles constituting the fluid. We do not even utilize kinetic equations to describe fluid motion. We describe such motion with a closure of these higher-level descriptions that we call the Navier–Stokes equation. As we delve further into our physical understanding of particle physics, we even discover that the 'exact' models that we studied in graduate school turn out to be approximations or closures of an ever more profound physics.

We shall now derive the evolution equations for the spectra, $U_i^x(t)$, $U_i^y(t)$ and $U_i^z(t)$, of the EDQNM closure when no mean fields are present.

From equations (9), we note that

$$\frac{\mathrm{d}\langle X_i(t)\rangle}{\mathrm{d}t} + \alpha_i\langle X_i(t))\rangle + \beta_i\langle Y_i(t)\rangle = \sum_{j,k=-N}^{N} c_{jki}\langle X_j(t)Y_k(t)\rangle \qquad \text{and} \qquad X \Longleftrightarrow Y. \tag{25}$$

We shall postulate the following condition on our ensemble averages of bilinear products for arbitrary values of j and k:

$$\langle X_j(t)X_k(t)\rangle = \delta_{jk}U_k^x(t) \qquad \langle Y_j(t)Y_k(t)\rangle = \delta_{jk}U_k^y(t) \qquad \langle X_j(t)Y_k(t)\rangle = \delta_{jk}U_k^z(t). \tag{26}$$

Ulitsky *et al* [7] have demonstrated that this approximation of 'random phase' appears to be well satisfied in turbulent Navier–Stokes fluids when no mean flow is present. (One should note that a certain subtlety is attached to the meaning of random phase in a finite ensemble [14].) This seems intuitively justifiable when there are many modes present due to the mixing produced by the nonlinear character of the equations. Furthermore, an absolute equilibrium ensemble for this dynamical system would also satisfy equation (26). One should also note that this random-phase approximation (RPA) is actually exact for the case of homogeneous turbulence, where the ensemble average of the product of two Fourier coefficients associated with wave vectors, $\boldsymbol{k}$ and $\boldsymbol{p}$, necessarily satisfies $\langle c^*(\boldsymbol{p})c(\boldsymbol{k})\rangle \propto \delta_{\boldsymbol{p},\boldsymbol{k}}$.

After implementing RPA in the absence of mean fields, equation (25) becomes

$$\frac{\mathrm{d}\langle(X_i(t))\rangle}{\mathrm{d}t} = \sum_{j=-N}^{N} c_{jji}U_j^z(t). \tag{27}$$

We shall impose the following additional symmetry condition on our coupling coefficients:

$$c_{jji} = -c_{-j-ji}. \tag{28}$$

(This condition is a cartoon version of the condition that we found was satisfied in our analysis of three-dimensional fluid turbulence in a slab [6]. We showed that the coupling coefficient, which we termed $g(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, satisfied $g(\boldsymbol{k}_+, \boldsymbol{p}_+, \boldsymbol{q}_+) = -g^*(\boldsymbol{k}_-, \boldsymbol{p}_-, \boldsymbol{q}_-)$, where $\boldsymbol{k}_-$ differed from $\boldsymbol{k}_+$ only in reversing the sign of the y-component of $\boldsymbol{k}_+$—the y-direction being the direction normal to the slab boundaries.)

We shall be confining our attention to ensembles for which $U_i^x(t) = U_{-i}^x(t)$, $U_i^y(t) = U_{-i}^y(t)$ and $U_i^z(t) = U_{-i}^z(t)$. One notes from equations (27) and (28) that our ensembles will not generate any mean fields, if none is initially present.

We now can use equations (9) to obtain evolution equations for bilinear and trilinear products of the coefficients. We find

$$\frac{1}{2}\frac{\mathrm{d}U_i^x(t)}{\mathrm{d}t} + \alpha_i U_i^x(t) + \beta_i U_i^z(t) = \sum_{j,k=-N}^{N} c_{jki}\langle X_i(t)X_j(t)Y_i(t)\rangle \qquad \text{and} \qquad X \Longleftrightarrow Y \tag{29}$$

$$\begin{aligned}\frac{\mathrm{d}(X_iX_jY_k)}{\mathrm{d}t} &+ (\alpha_i+\alpha_j+\alpha_k)X_iX_jY_k + \beta_iY_iX_jY_k + \beta_jY_jX_iY_k + \beta_kX_kX_jX_i \\ &= \sum_{l,m=-N}^{N}(c_{lmi}X_lY_mX_jY_k + c_{lmj}X_lY_mX_iY_k + c_{lmk}Y_lX_mX_iX_j)\end{aligned} \tag{30}$$

where we suppressed the time dependence of the spectral coefficients in the latter equation.

Now we make the standard 'eddy-damped Markovian' approximation. Instead of integrating equation (30) to obtain the time dependence of the triple correlation, the Markovian approximation is made that the triple correlation on the left is merely proportional to the right-hand side of the equation. The proportionality is turned into an equality with the use of an 'eddy-damping' function, θ, a phenomenologically chosen function that is assumed never to be negative. This yields the eddy-damped Markovian approximation,

$$\langle X_iX_jY_k\rangle_t = \theta_{ikj}^x(t)\sum_{l,m=-N}^{N}(c_{lmi}\langle X_lY_mX_jY_k\rangle + c_{lmj}\langle X_lY_mX_iY_k\rangle + c_{lmk}\langle Y_lX_mX_iX_j\rangle)_t \tag{31}$$

and a corresponding equation with all the x's and y's exchanged. Note that, without loss of generality, the θ's can be defined to be symmetric under the exchange of their first and third subscripts, i.e.

$$\theta_{ikj}^x(t) = \theta_{jki}^x(t) \qquad \theta_{ikj}^y(t) = \theta_{jki}^y(t). \tag{32}$$

We now make the quasinormal approximation that

$$\langle A_iA_jA_kA_l\rangle = \langle A_iA_j\rangle\langle A_kA_l\rangle + \langle A_iA_k\rangle\langle A_jA_l\rangle + \langle A_iA_l\rangle\langle A_jA_k\rangle \tag{33}$$

where the A's represent any of the X's or Y's. This is termed *quasi*normal because, although equation (33) is what one would expect, if the spectral coefficients were governed by a normal distribution, we note from the equation just above that this moment produces a triple correlation, which would be absent from any normal distribution.

Inserting both equations (31) and (33) into equations (29) and implementing equations (11) and (32), as well as the random-phase approximation, equation (26), we obtain the EDQNM closure:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}U_i^x}{\mathrm{d}t} + \alpha_iU_i^x + \beta_iU_i^z &= \sum_{j,k=-N}^{N}\theta_{jki}^x c_{jki}^2 U_k^y\left(U_j^x - U_i^x\right) + \sum_{j,k=-N}^{N}\theta_{jki}^x c_{jki}c_{kji}\left(U_k^zU_j^z - U_j^xU_i^z\right) \\ &+ \sum_{j,k=-N}^{N}\theta_{jki}^x c_{jki}c_{kij}\left(U_k^zU_i^z - U_i^xU_j^z\right) \\ &+ \sum_{j,k=-N}^{N}\theta_{ijj}^x c_{jji}c_{kki}U_k^zU_j^z + \sum_{j,k=-N}^{N}\theta_{iij}^x c_{jii}c_{kkj}U_k^zU_i^z.\end{aligned} \tag{34}$$

The corresponding equation for the evolution of U_i^y is obtained by interchanging all x's and y's. In a similar fashion, we can derive an equation for the evolution of U_i^z:

$$\frac{\mathrm{d}U_i^z}{\mathrm{d}t} + 2\alpha_i U_i^z + \beta_i\left(U_i^x + U_i^y\right) = \Bigg\{\Bigg[\sum_{j,k=-N}^{N} \theta_{ijk}^x c_{jki}^2\left(U_k^z U_j^z - U_i^z U_k^x\right) + \sum_{j,k=-N}^{N} \theta_{ijk}^x c_{jki} c_{kji}\left(U_k^x - U_i^x\right)U_j^y + \sum_{j,k=-N}^{N} \theta_{ijk}^x c_{jki} c_{kij}\left(U_k^z U_i^x - U_j^z U_i^z\right) + \sum_{j,k,=-N}^{N} \theta_{ijj}^x c_{jji} c_{kki} U_k^z U_j^z + \sum_{j,k,=-N}^{N} \theta_{iji}^x c_{jii} c_{kkj} U_k^z U_i^x\Bigg] + [x \Longleftrightarrow y]\Bigg\}. \quad (35)$$

Suppose that initially, $U_i^z(0) = U_{-i}^z(0)$, for arbitrary i. To maintain the absence of mean fields throughout the evolution, we must maintain this symmetry in time. To do so, we shall require that initially:

$$U_i^x(0) = U_{-i}^x(0) \qquad \text{and} \qquad U_i^y(0) = U_{-i}^y(0) \qquad \forall\, i. \quad (36)$$

Since the evolution equations, equations (34) and (35), are quadratic in the coupling constants, this initial symmetry will be propagated in time if we require

$$\theta_{ijk}^x = \theta_{-i-j-k}^x \qquad \theta_{ijk}^y = \theta_{-i-j-k}^y \quad (37)$$

as well as the additional symmetry:

$$c_{ijk} = s\, c_{-i-j-k} \qquad \forall\, i, j, k \quad (38)$$

where s is a subscript-independent constant equal to either +1 or −1. We shall impose this symmetry on our coupling constants and assume that all of the U's are initially invariant under sign-reversal of their subscripts. The evolution equations then reduce to:

$$\frac{1}{2}\frac{\mathrm{d}U_i^x}{\mathrm{d}t} + \alpha_i U_i^x + \beta_i U_i^z = \sum_{j,k=-N}^{N} \theta_{jki}^x c_{jki}^2 U_k^y\left(U_j^x - U_i^x\right) + \sum_{j,k=-N}^{N} \theta_{jki}^x c_{jki} c_{kji}\left(U_k^z U_j^z - U_j^x U_i^z\right) + \sum_{j,k=-N}^{N} \theta_{jki}^x c_{jki} c_{kij}\left(U_k^z U_i^z - U_i^x U_j^z\right) \quad (39)$$

$$\frac{1}{2}\frac{\mathrm{d}U_i^y}{\mathrm{d}t} + \alpha_i U_i^y + \beta_i U_i^z = \sum_{j,k=-N}^{N} \theta_{jki}^y c_{jki}^2 U_k^x\left(U_j^y - U_i^y\right) + \sum_{j,k=-N}^{N} \theta_{jki}^y c_{jki} c_{kji}\left(U_k^z U_j^z - U_j^y U_i^z\right) + \sum_{j,k=-N}^{N} \theta_{jki}^y c_{jki} c_{kij}\left(U_k^z U_i^z - U_i^y U_j^z\right) \quad (40)$$

$$\frac{\mathrm{d}U_i^z}{\mathrm{d}t} + 2\alpha_i U_i^z + \beta_i\left(U_i^x + U_i^y\right) = \Bigg\{\Bigg[\sum_{j,k=-N}^{N} \theta_{ijk}^x c_{jki}^2\left(U_k^z U_j^z - U_i^z U_k^x\right) + \sum_{j,k=-N}^{N} \theta_{ijk}^x c_{jki} c_{kji}\left(U_k^x - U_i^x\right)U_j^y + \sum_{j,k=-N}^{N} \theta_{ijk}^x c_{jki} c_{kij}\left(U_k^z U_i^x - U_j^z U_i^z\right)\Bigg] + [x \Longleftrightarrow y]\Bigg\}. \quad (41)$$

Before concluding this section, we shall show that the ideal MHD invariants remain invariant under this closure. We shall use the general evolution equation for U_i^x, equation (34), to demonstrate that $\sum_{i=-N}^{N} U_i^x$ is conserved in the lossless limit in which all of the α_i's and all of the β_i's are set equal to zero. We wish to examine $\sum_{i=-N}^{N} \frac{\mathrm{d}U_i^x(t)}{\mathrm{d}t}$. There will be five sums on the right-hand side over dummy indices, i, j and k. Consider the first sum by itself. By virtue of equations (11) and (32), the summand is antisymmetric under the interchange of i and j. Hence this sum vanishes. Now consider the second and third sums jointly. If one interchanges the dummy indices, i and j, everywhere in the third summand and makes use of equations (11) and (32), one observes that it will precisely cancel the second summand. Finally, similar interchange of the i and j dummy indices in the fifth summand and application of equations (11) and (32) demonstrate that it will cancel the fourth summand. Of course, $\sum_{i=-N}^{N} U_i^y$ will also be conserved since its evolution equation has precisely the same structure. We thus conclude that this closure preserves the two ideal MHD invariants, total energy and total cross-helicity.

5. Demonstration of realizability of the spectra in the absence of mean fields

In this section, we shall demonstrate that if the spectra initially satisfy the necessary physical conditions for arbitrary k,

$$U_k^x(0) \geqslant 0 \qquad U_k^y(0) \geqslant 0 \qquad \text{and} \qquad \left|U_k^z(0)\right| \leqslant \left[U_k^x(0)U_k^y(0)\right]^{\frac{1}{2}} \tag{42}$$

then as these spectra evolve under the EDQNM equations, equations (39), (40) and (41), they will continue to maintain these properties for all time.

Suppose some initially positive spectral element of U^x or U^y were to become negative. Let us assume that there is a first element, U_i^x, to do so, at time t_0. Then at this time, it must have a negative time-derivative. (We are going to refrain in this paper from considering more special cases such as t_0 being a point of inflection.) We shall also assume that there has been no violation of the Schwarz inequality, the third property listed just above. Let us evaluate this derivative using equation (39), and observing that if $U_i^x(t_0) = 0$, then necessarily also $U_i^z(t_0) = 0$. We find:

$$\frac{1}{2}\frac{\mathrm{d}U_i^x}{\mathrm{d}t_0} = \sum_{j,k=-N}^{N} \left[\theta_{jki}^x c_{jki}^2 U_k^y U_j^x + \theta_{jki}^x c_{jki} c_{kji} U_k^z U_j^z\right]_{t_0}. \tag{43}$$

For arbitrary values of k, we may define a quantity, $\lambda_k(t_0)$, by:

$$\lambda_k(t_0) = \frac{U_k^z(t_0)}{\left[U_k^x(t_0)U_k^y(t_0)\right]^{\frac{1}{2}}}. \tag{44}$$

The Schwarz inequality then guarantees that

$$|\lambda_k(t_0)| \leqslant 1. \tag{45}$$

We can then rewrite the above expression as

$$\frac{1}{2}\frac{\mathrm{d}U_i^x}{\mathrm{d}t_0} = \sum_{j,k=-N}^{N} \theta_{jki}^x \left[c_{jki}^2 U_k^y U_j^x + c_{jki}c_{kji}\lambda_k\lambda_j\left(U_k^x U_j^x U_k^y U_j^y\right)^{\frac{1}{2}}\right]_{t_0}$$

$$= \sum_{j,k=-N}^{N} \theta_{jki}^x \left\{c_{jki}^2\left(1-\lambda_j^2\right)U_k^y U_j^x + \frac{1}{2}\left[\lambda_j c_{jki}\left(U_k^y U_j^x\right)^{\frac{1}{2}} + \lambda_k c_{kji}\left(U_j^y U_k^x\right)^{\frac{1}{2}}\right]^2\right\}_{t_0}. \tag{46}$$

Using equation (45) and the given positive nature of all of the spectral elements of U^x and U^y on the right-hand side of this equation, we conclude that

$$\frac{\mathrm{d}U_i^x(t_0)}{\mathrm{d}t_0} \geqslant 0. \tag{47}$$

This contradicts our assumption that U_i^x becomes negative at time t_0. In a similar manner, we can also show that none of the spectral elements of U^y can become negative at any time.

We shall again suppose that equations (42) are satisfied and now demonstrate that the last of these, the Schwarz inequality, is also maintained for all time. Suppose that there is an element, say the ith element, that first violates this inequality and does so at time t_s. Then at this time, we would find that

$$\frac{\left[U_i^z(t_s)\right]^2}{U_i^x(t_s)U_i^y(t_s)} = 1 \tag{48}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\left[U_i^z(t_s)\right]^2}{U_i^x(t_s)U_i^y(t_s)}\right\} > 0. \tag{49}$$

Observe that

$$\left\{\frac{U_i^x(t_s)U_i^y(t_s)}{\left[U_i^z(t_s)\right]^2}\right\}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\left[U_i^z(t_s)\right]^2}{U_i^x(t_s)U_i^y(t_s)}\right\} = \frac{2\dot{U}_i^z(t_s)}{U_i^z(t_s)} - \frac{\dot{U}_i^x(t_s)}{U_i^x(t_s)} - \frac{\dot{U}_i^y(t_s)}{U_i^y(t_s)}. \tag{50}$$

We now insert equations (34) and (35) to evaluate the time derivatives on the right-hand side:

$$\begin{aligned}
\frac{2\dot{U}_i^z(t_s)}{U_i^z(t_s)} - \frac{\dot{U}_i^x(t_s)}{U_i^x(t_s)} - \frac{\dot{U}_i^y(t_s)}{U_i^y(t_s)} &= -4\alpha_i - 2\beta_i\left[\frac{U_i^x(t_s)}{U_i^z(t_s)} + \frac{U_i^y(t_s)}{U_i^z(t_s)}\right] \\
&+ \Bigg\{\Bigg[2\sum_{j,k=-N}^{N}\theta_{ijk}^x c_{jki}^2\left(\frac{U_k^zU_j^z}{U_i^z} - U_k^x\right) + 2\sum_{j,k=-N}^{N}\theta_{ijk}^x c_{jki}c_{kji}\left(\frac{U_k^x}{U_i^z} - \frac{U_i^x}{U_i^z}\right)U_j^y \\
&+ 2\sum_{j,k=-N}^{N}\theta_{ijk}^x c_{jki}c_{kij}\left(\frac{U_k^zU_i^x}{U_i^z} - U_j^z\right) + 2\sum_{j,k=-N}^{N}\theta_{ijj}^x c_{jji}c_{kki}\frac{U_k^zU_j^z}{U_i^z} \\
&+ 2\sum_{j,k=-N}^{N}\theta_{iji}^x c_{jii}c_{kkj}\frac{U_k^zU_i^x}{U_i^z} + 2\alpha_i + 2\beta_i\frac{U_i^z}{U_i^x} \\
&- 2\sum_{j,k=-N}^{N}\theta_{jki}^x c_{jki}^2\left(\frac{U_j^x}{U_i^x} - 1\right)U_k^y - 2\sum_{j,k=-N}^{N}\theta_{jki}^x c_{jki}c_{kji}\left(\frac{U_k^zU_j^z}{U_i^x} - \frac{U_j^xU_i^z}{U_i^x}\right) \\
&- 2\sum_{j,k=-N}^{N}\theta_{jki}^x c_{jki}c_{kij}\left(\frac{U_k^zU_i^z}{U_i^x} - U_j^z\right) - 2\sum_{j,k=-N}^{N}\theta_{ijj}^x c_{jji}c_{kki}\frac{U_k^zU_j^z}{U_i^x} \\
&- 2\sum_{j,k=-N}^{N}\theta_{iij}^x c_{jii}c_{kkj}\frac{U_k^zU_i^z}{U_i^x}\Bigg] + [x \Longleftrightarrow y]\Bigg\}_{t_s}.
\end{aligned} \tag{51}$$

We now must impose two new symmetry conditions on the eddy-damping functions:

$$\theta^x = \theta^y = \theta \tag{52}$$

$$\theta_{ijk} = \theta_{jik}. \tag{53}$$

Taking into account the already existing symmetry stated in equation (32), equation (52) implies that the eddy-damping function is totally symmetric under the exchange of any two of its subscripts. Observing from equation (48) that at time t_s

$$\frac{U_i^x}{U_i^z} - \frac{U_i^z}{U_i^y} = \frac{U_i^y}{U_i^z} - \frac{U_i^z}{U_i^x} = 0 \tag{54}$$

we can simplify equation (51):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_s}\left[\frac{\left(U_i^z\right)^2}{U_i^x U_i^y}\right] = \Bigg\{ & 2\sum_{j,k=-N}^{N} \theta_{ijk} U_k^z U_j^z \left[\frac{2c_{jki}^2}{U_i^z} - c_{jki}c_{kji}\left(\frac{1}{U_i^x} + \frac{1}{U_i^y}\right)\right] \\ & + 2\sum_{j,k=-N}^{N} \theta_{ijk} U_k^x U_j^y \left(\frac{2c_{jki}c_{kji}}{U_i^z} - \frac{c_{jki}^2}{U_i^y} - \frac{c_{kji}^2}{U_i^x}\right) \\ & + 2\sum_{j,k=-N}^{N} \theta_{ijj} U_k^z U_j^z c_{jji} c_{kki}\left(\frac{2}{U_i^z} - \frac{1}{U_i^x} - \frac{1}{U_i^y}\right)\Bigg\}_{t_s}.\end{aligned} \tag{55}$$

No mean fields are present in the case under consideration. From the discussion following equation (28), the final sum vanishes, since $\sum_k c_{kki} U_k^z = 0$.

We again use equations (44) and (45), as well as $\lambda_i = \lambda_i^{-1} = \pm 1$ implied by equation (48), to rewrite equation (55) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_s}\left[\frac{\left(U_i^z\right)^2}{U_i^x U_i^y}\right] = \Bigg\{ & 2\sum_{j,k=-N}^{N} \theta_{ijk}\lambda_k\lambda_j \left(U_k^x U_k^y U_j^x U_j^y\right)^{\frac{1}{2}} \left[\frac{2c_{jki}^2\lambda_i}{\left(U_i^x U_i^y\right)^{\frac{1}{2}}} - c_{jki}c_{kji}\left(\frac{1}{U_i^x} + \frac{1}{U_i^y}\right)\right] \\ & + 2\sum_{j,k=-N}^{N} \theta_{ijk} U_k^x U_j^y \left(\frac{2c_{jki}c_{kji}\lambda_i}{\left(U_i^x U_i^y\right)^{\frac{1}{2}}} - \frac{c_{jki}^2}{U_i^y} - \frac{c_{kji}^2}{U_i^x}\right)\Bigg\}_{t_s} \\ = \Bigg\{ & 2\sum_{j,k=-N}^{N} \theta_{ijk}\lambda_k\lambda_j \left(U_k^x U_k^y U_j^x U_j^y\right)^{\frac{1}{2}} \left[\frac{2c_{jki}^2\lambda_i}{\left(U_i^x U_i^y\right)^{\frac{1}{2}}} - c_{jki}c_{kji}\left(\frac{1}{U_i^x} + \frac{1}{U_i^y}\right)\right] \\ & - 2\sum_{j,k=-N}^{N} \theta_{ijk} U_k^x U_j^y \left[\frac{c_{jki}}{\left(U_i^y\right)^{\frac{1}{2}}} - \frac{\lambda_i c_{kji}}{\left(U_i^x\right)^{\frac{1}{2}}}\right]^2\Bigg\}_{t_s}.\end{aligned} \tag{56}$$

We wish to prove that this quantity is not positive. If we make the total of the two sums as large as possible and still find that it is not positive, then we shall be certain that the sum is never positive. The second sum is clearly not positive. Each term in the first sum has an overall factor of $\lambda_k\lambda_j$. We shall make the first sum as large as possible by letting this factor have its largest possible magnitude, unity! Accordingly, we shall choose this factor for each choice of j and k such that the associated summand will be positive. We shall define:

$$e_{jk} \equiv \lambda_k\lambda_j \tag{57}$$

so that

$$e_{jk}^2 = \lambda_i^2 = 1. \tag{58}$$

We then observe that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t_s}\left[\frac{\left(U_i^z\right)^2}{U_i^x U_i^y}\right] \leqslant \Bigg(& 2\sum_{j,k=-N}^{N} \theta_{ijk} e_{jk} \left(U_k^x U_k^y U_j^x U_j^y\right)^{\frac{1}{2}} \left[\frac{2c_{jki}^2\lambda_i}{\left(U_i^x U_i^y\right)^{\frac{1}{2}}} - c_{jki}c_{kji}\left(\frac{1}{U_i^x}+\frac{1}{U_i^y}\right)\right] \\
& + 2\sum_{j,k=-N}^{N} \theta_{ijk} U_k^x U_j^y \left(\frac{2c_{jki}c_{kji}\lambda_i}{\left(U_i^x U_i^y\right)^{\frac{1}{2}}} - \frac{c_{jki}^2}{U_i^y} - \frac{c_{kji}^2}{U_i^x}\right) \\
= & -2\sum_{j,k=-N}^{N} \theta_{ijk} \left[U_k^x U_j^y \left(\frac{c_{jki}^2}{U_i^y}+\frac{c_{kji}^2}{U_i^x}\right) - 2\lambda_i e_{jk} c_{jki}^2 \left(\frac{U_k^x U_k^y U_j^x U_j^y}{U_i^x U_i^y}\right)^{\frac{1}{2}}\right] \\
& - 2\sum_{j,k=-N}^{N} \theta_{ijk} c_{jki} c_{kji} \left[e_{jk}\left(U_k^x U_k^y U_j^x U_j^y\right)^{\frac{1}{2}}\left(\frac{1}{U_i^x}+\frac{1}{U_i^y}\right) - 2\lambda_i \frac{U_k^x U_j^y}{\left(U_i^x U_i^y\right)^{\frac{1}{2}}}\right] \\
= & -2\sum_{j,k=-N}^{N} \theta_{ijk} \left\{\left[\left(\frac{U_k^x U_j^y}{U_i^y}\right)^{\frac{1}{2}} - \lambda_i e_{jk}\left(\frac{U_j^x U_k^y}{U_i^x}\right)^{\frac{1}{2}}\right] c_{jki}\right\}^2 \\
& - 2\sum_{j,k=-N}^{N} \theta_{ijk} e_{jk} \left\{\left[\left(\frac{U_k^x U_j^y}{U_i^y}\right)^{\frac{1}{2}} - \lambda_i e_{jk}\left(\frac{U_j^x U_k^y}{U_i^x}\right)^{\frac{1}{2}}\right] c_{jki}\right\} \\
& \times \left\{\left[\left(\frac{U_j^x U_k^y}{U_i^y}\right)^{\frac{1}{2}} - \lambda_i e_{jk}\left(\frac{U_k^x U_j^y}{U_i^x}\right)^{\frac{1}{2}}\right] c_{kji}\right\} \\
= & -\sum_{j,k=-N}^{N} \theta_{ijk} \left\{\left[\left(\frac{U_k^x U_j^y}{U_i^y}\right)^{\frac{1}{2}} - \lambda_i e_{jk}\left(\frac{U_j^x U_k^y}{U_i^x}\right)^{\frac{1}{2}}\right] c_{jki}\right. \\
& \left. + e_{jk}\left[\left(\frac{U_j^x U_k^y}{U_i^y}\right)^{\frac{1}{2}} - \lambda_i e_{jk}\left(\frac{U_k^x U_j^y}{U_i^x}\right)^{\frac{1}{2}}\right] c_{kji}\right\}^2 \Bigg)_{t_s} \leqslant 0.
\end{aligned}
\qquad (59)
$$

But this violates our assumption, equation (49). Thus the Schwarz inequality is maintained by the EDQNM closure for the evolution of the spectra of MHD turbulence.

6. Summary

The so-called EDQNM closure that has been used in the study of Navier–Stokes turbulence has been examined for its applicability to general MHD turbulence. The MHD equations provide a more stringent test on the usefulness of the EDQNM closure because in MHD the demands of realizability are more severe. Not only must the components of the turbulent kinetic energy spectrum be positive semidefinite, but also the components of the turbulent magnetic energy spectrum must be positive semidefinite. Furthermore, the components of the turbulent cross-helicity spectrum must satisfy a Schwarz inequality with respect to the components of the turbulent kinetic and magnetic energy spectra.

Because the demands of the notation involved in using the full MHD equations would have diverted attention away from the essential details of analysing the applicability of the EDQNM closure to MHD, we developed a stripped-down version, or toy model, of MHD in which only the essential elements of symmetry were kept. In this study, we also assumed for reasons of simplicity that the magnetic fields and the velocity fields could be represented by the same set of solenoidal basis vectors. This need not be the case. The usefulness of the EDQNM closure remains to be explored for such cases.

We found that when certain symmetry conditions are placed on the eddy-damping functions employed, an EDQNM closure does produce evolution equations that yield realizable spectra. However, it seems to us that the required symmetry conditions are unduly severe; whether the closure according to equation (31) is performed on $\langle X_i Y_j X_k \rangle$ or on $\langle Y_i X_j Y_k \rangle$, the same eddy-damping function must be used and additionally be symmetric under exchange of any two of its three subscripts!

Had we instead worked with the primordial MHD equations that specify $\partial \boldsymbol{v}/\partial t$ and $\partial \boldsymbol{B}/\partial t$ and obtained their implications for the evolution of the triple correlations, $\langle v_i v_j v_k \rangle$, $\langle v_i v_j B_k \rangle$, $\langle v_i B_j B_k \rangle$ and $\langle B_i B_j B_k \rangle$, we would have found that all of the linear terms in each of the evolution equations involved only one of the triple correlations in contrast to equation (31). Nevertheless, had we gone to demand positivity of the spectral components of the turbulent magnetic and kinetic energies, as well as a Schwarz inequality on the components of the turbulent cross-helicity, we would again have found a highly restricted eddy-damping function. Its limitations might be better understood if one were to develop a more fundamental closure for MHD, such as a direct-interaction approximation [15] or a test-field model [16].

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